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NOTE ON THE UNIQUENESS OF THE SINK IN SYMBIOSIS SYSTEM

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Abstract

We show that there is a counter-example for the uniqueness of the sink in symbiosis system under assumptions given in [1]. We also give a suitable condition for the uniqueness of the sink.

1. Introduction

Let x and y denote the populations of two species. The generalized equations of growth of the two populations are written in the form

$$\begin{cases} x' = M(x, y)x, \\ y' = N(x, y)y, \end{cases} \tag{1}$$

where the growth rates M and N are functions of both variables. We assume that x and y are nonnegative, and M and N are sufficiently smooth. We consider the case that x and y are in symbiosis, that is, we make the following assumptions on M and N:

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(H1) An increase of either population leads to an increase in the growth rate of the other. Hence,

$$M_y(x, y) > 0, \quad \text{and} \quad N_x(x, y) > 0,$$

where $M_y = \partial M / \partial y$ and $N_x = \partial N / \partial x$.

(H2) The total food supply is limited. Hence, for some $A > 0$, $B > 0$, we have

$$M(x, y) < 0, \quad \text{for} \quad x > A,$$

$$N(x, y) < 0, \quad \text{for} \quad y > B.$$

(H3) If both populations are very small, then they both increase. Hence,

$$M(0, 0) > 0, \quad \text{and} \quad N(0, 0) > 0.$$

We also assume that the intersections of the curves $M^{-1}(0)$ and $N^{-1}(0)$ are finite, and that all are transverse.

Under these assumptions, the analysis of (1) is suggested as an exercise in [1] such that 'show that, if $M_x(x, y) < 0$ and $N_y(x, y) < 0$, there is a unique sink Z, and Z is the ω-limit set for all (x, y) with $x > 0$, $y > 0$.' However, there is a counter-example for the uniqueness of the sink under such assumptions. The purpose of this paper is to show such a counter-example, and give a sufficient condition for the uniqueness of the sink.

2. Counter-Example for the Uniqueness of the Sink

Define $M(x, y)$ and $N(x, y)$ as

$$\begin{cases} M(x, y) = 1 - x + \dfrac{6}{1 + e^{-(y-4)}}, \\ N(x, y) = 1 + \dfrac{6}{1 + e^{-(x-4)}} - y. \end{cases} \tag{2}$$

1. We have

$$M_y(x, y) = \frac{6e^{-(y-4)}}{(1+e^{-(y-4)})^2} = \frac{6e^{y+4}}{(e^y + e^4)^2} > 0,$$

$$N_x(x, y) = \frac{6e^{-(x-4)}}{(1+e^{-(x-4)})^2} = \frac{6e^{x+4}}{(e^x + e^4)^2} > 0.$$

2. Since $\frac{6}{1+e^{-(y-4)}} < 6$ and $\frac{6}{1+e^{-(x-4)}} < 6$, we have

$$M(x, y) < 0 \quad \text{for} \quad x > 7, \quad \text{and} \quad N(x, y) < 0 \quad \text{for} \quad y > 7.$$

3. Since $1 + e^4 > 6$, we have

$$M(0, 0) = 1 - \frac{6}{1+e^4} > 0, \quad \text{and} \quad N(0, 0) = 1 - \frac{6}{1+e^4} > 0.$$

Also, we have

$$M_x(x, y) = -1 < 0, \quad \text{and} \quad N_y(x, y) = -1 < 0.$$

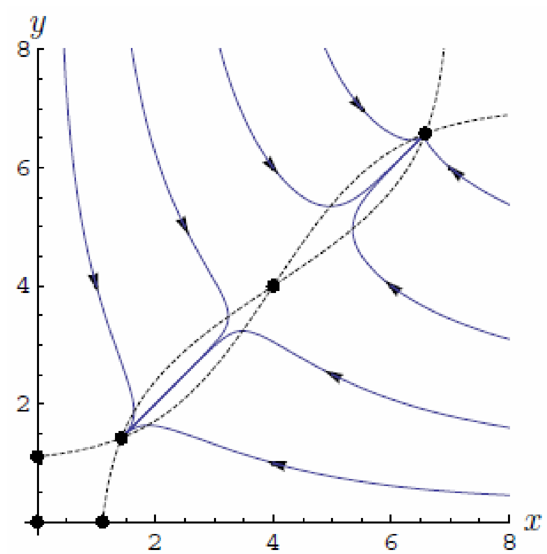

Figure 1. The phase portrait of (1) with (2). The dotted curves represent $\mu = M^{-1}(0)$ and $\nu = N^{-1}(0)$.

Thus, all assumptions on M and N given in [1] for the uniqueness of the sink are established. However, the phase portrait is given by Figure 1, which shows that there exist two sinks. Note that, the dotted curves represent $\mu = M^{-1}(0)$ and $\nu = N^{-1}(0)$.

The intersections of μ and ν are given by roots of $M(x, x) = N(x, x) = 0$. We have

$$M(1, 1) = \frac{6}{1+e^3} > 0,$$

$$M(2, 2) = -1 + \frac{6}{1+e^2} = \frac{5-e^2}{1+e^2} < 0,$$

$$M(4, 4) = -3 + \frac{6}{1+e^0} = 0,$$

$$M(6, 6) = -5 + \frac{6}{1+e^{-2}} = \frac{e^2-5}{1+e^2} > 0,$$

$$M(7, 7) = -6 + \frac{6}{1+e^{-3}} = -\frac{6}{1+e^3} < 0.$$

Therefore, there exist at least three equilibrium $(x_1, x_1), (x_2, x_2), (x_3, x_3)$ such that

$$1 < x_1 < 2 < x_2 = 4 < 6 < x_3 < 7.$$

By the implicit function theorem, we can express $\mu = M^{-1}(0)$ by $y_1(x)$ such that

$$y_1'(x) = -\frac{M_x}{M_y}, \; y_1''(x) = -\frac{M_{xx}M_y^2 - 2M_{xy}M_xM_y + M_{yy}M_x^2}{M_y^3}.$$

Also, $\nu = N^{-1}(0)$ is given by $y_2(x)$ such that

$$y_2'(x) = -\frac{N_x}{N_y}, \; y_2''(x) = -\frac{N_{xx}N_y^2 - 2N_{xy}N_xN_y + N_{yy}N_x^2}{N_y^3}.$$

Here, we have

$$M_{xx}(x,\ y) = M_{xy}(x,\ y) = 0, \quad M_{yy}(x,\ y) = -\frac{6e^{y+4}(e^{y} - e^{4})}{(e^{y} + e^{4})^{3}},$$

and

$$N_{xx}(x,\ y) = -\frac{6e^{x+4}(e^{x} - e^{4})}{(e^{x} + e^{4})^{3}}, \quad N_{xy}(x,\ y) = N_{yy}(x,\ y) = 0.$$

Thus, we have

$$y_1'(x) = \frac{(e^{y} + e^{4})^{2}}{6e^{y+4}} > 0, \quad y_1''(x) = \frac{1}{36}e^{-2(y+4)}(e^{y} - e^{4})\,(e^{y} + e^{4})^{3},$$

and

$$y_2'(x) = \frac{6e^{x+4}}{(e^{x} + e^{4})^{2}} > 0, \quad y_2''(x) = -\frac{6e^{x+4}(e^{x} - e^{4})}{(e^{x} + e^{4})^{3}}.$$

Now, we know that if $x < 4$ and $y < 4$, then $y_1''(x) < 0$ and $y_2''(x) > 0$. This implies that $y_1'(x_1) > y_2'(x_1)$, because $y_1'(x_1) < y_2'(x_1)$ contradicts that $y_1(4) = y_2(4)$. Hence, we have

$$-\frac{M_x(x_1,\ x_1)}{M_y(x_1,\ x_1)} > -\frac{N_x(x_1,\ x_1)}{N_y(x_1,\ x_1)}.$$

The Jacobian matrix at $(x_1,\ x_1)$ is given by

$$A = \begin{pmatrix} x_1 M_x(x_1,\ x_1) & x_1 M_y(x_1,\ x_1) \\ x_1 N_x(x_1,\ x_1) & x_1 N_y(x_1,\ x_1) \end{pmatrix}.$$

We have $\mathrm{tr}A = x_1 M_x + x_1 N_y < 0$, and $\det A = x_1^2(M_x N_y - M_y N_x) > 0$. Thus, we have that $(x_1,\ x_1)$ is a sink.

In the same way, we can show that $(x_3,\ x_3)$ is a sink, too. Therefore, there exist at least two sinks for (1) with (2).

3. Sufficient Condition of the Uniqueness of the Sink

Theorem 1. *Assume that* $M(x, y)$ *and* $N(x, y)$ *satisfy* (H1)-(H3). *Moreover, we make the following additional assumptions on* M *and* N:

(H4) $M_x(x, y) < 0$ *and* $N_y(x, y) < 0$.

(H5) $M_{xx}(x, y) \leq 0$, $M_{xy}(x, y) \leq 0$, *and* $M_{yy}(x, y) \leq 0$.

(H6) $N_{xx}(x, y) \leq 0$, $N_{xy}(x, y) \leq 0$, *and* $N_{yy}(x, y) \leq 0$.

Then (1) *has a unique sink Z, and Z is the* ω*-limit set for all* (x, y) *with* $x > 0$, $y > 0$.

Proof. Using the implicit function theorem, we can express $\mu = M^{-1}(0)$ by $y_1(x)$ such that

$$y_1'(x) = -\frac{M_x}{M_y}, \quad y_1''(x) = -\frac{M_{xx}M_y^2 - 2M_{xy}M_xM_y + M_{yy}M_x^2}{M_y^3}.$$

From (H1), (H4), and (H5), we have $y_1'(x) > 0$ and $y_1''(x) \geq 0$ for all $x \geq 0$. Hence, $y_1(x)$ is downwards convex. Also, we can express $\nu = N^{-1}(0)$ by $y_2(x)$ such that

$$y_2'(x) = -\frac{N_x}{N_y}, \quad y_2''(x) = -\frac{N_{xx}N_y^2 - 2N_{xy}N_xN_y + N_{yy}N_x^2}{N_y^3}.$$

From (H1), (H4), and (H6), we have $y_2'(x) > 0$ and $y_2''(x) \leq 0$ for all $x \geq 0$. Hence, $y_2(x)$ is upwards convex.

From (H2) and (H3), there exist a_1 and a_2 such that $0 < a_1 < a_2 \leq A$, $y_1(a_1) = 0$, and $\lim_{x \to a_2} y_1(x) = \infty$. Also, from (H2) and (H3), there exists b such that $0 < b < B$, $y_1(0) = b$, and $y_1(x) < B$. Therefore, there exists a unique intersection point (x^*, y^*) of the curves μ and ν such that

$$y_1'(x^*) > y_2'(x^*) > 0.$$

This implies that

$$-\frac{M_x(x^*, y^*)}{M_y(x^*, y^*)} > -\frac{N_x(x^*, y^*)}{N_y(x^*, y^*)}.$$

The Jacobian matrix at (x^*, y^*) is given by

$$A = \begin{pmatrix} x^* M_x(x^*, y^*) & x^* M_y(x^*, y^*) \\ y^* N_x(x^*, y^*) & y^* N_y(x^*, y^*) \end{pmatrix}.$$

Then, we have $\mathrm{tr} A = x^* M_x + y^* N_y < 0,$ and $\det A = x^* y^* (M_x N_y - M_y N_x) > 0.$ Thus, we have that (x^*, y^*) is the sink. □

Example. Define $M(x, y)$ and $N(x, y)$ as

$$\begin{cases} M(x, y) = 1 - x + 3(1 - e^{-y}), \\ N(x, y) = 1 + 3(1 - e^{-x}) - y. \end{cases} \tag{3}$$

We can easily confirm the assumptions (H1)-(H6) of Theorem 1 as follows.

1. $M_y(x, y) = 3e^{-y} > 0,\ N_x(x, y) = 3e^{-x} > 0.$

2. $M(x, y) < 0$ for $x > 0$, and $N(x, y) < 0$ for $y > 4$.

3. $M(0, 0) = 1 > 0,\ N(0, 0) = 1 > 0.$

4. $M_x(x, y) = -1 < 0,\ N_y(x, y) = -1 < 0.$

5. $M_{xx}(x, y) = 0,\ M_{xy}(x, y) = 0,$ *and* $M_{yy}(x, y) = -3e^{-y} \leq 0.$

6. $N_{xx}(x, y) = -3e^{-x} \leq 0,\ N_{xy}(x, y) = 0,$ *and* $N_{yy}(x, y) = 0.$

Therefore, Theorem 1 implies that there exists a unique sink for (1) with (3). The phase portrait is given by Figure 2 in which the dotted curves represent $\mu = M^{-1}(0)$ and $\nu = N^{-1}(0)$. This figure shows that there exists a unique sink. □

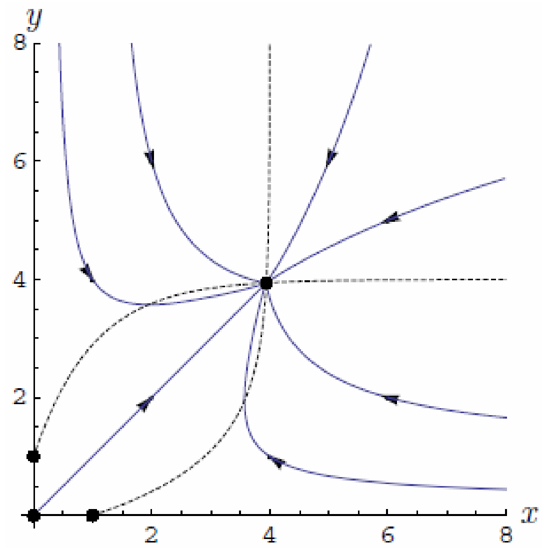

Figure 2. The phase portrait of (1) with (3). The dotted curves represent $\mu = M^{-1}(0)$ and $\nu = N^{-1}(0)$.

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